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Equilibria in games with weak payoff externalities*

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Abstract

We study pure strategy Nash equilibria in games with *partial* weak payoff externalities, which generalize games with weak payoff externalities introduced by Ania (2008). For an n-person symmetric game with partial weak payoff externalities, we show that the set of symmetric Nash equilibria coincides with the set of symmetric evolutionary equilibria (Schaffer, 1989), and that if the game is finite, the set of symmetric Nash equilibria, which are equivalent to symmetric evolutionary equilibria, is nonempty whenever strategies are linearly ordered and the payoff function is quasiconcave. The existence result generalizes that of Iimura and Watanabe (2016), since, if it is symmetric, a weakly unilaterally competitive game (Kats and Thisse, 1992) is one with partial weak payoff externalities.

Keywords: existence of equilibrium, symmetric evolutionary equilibrium, games with weak payoff externalities, weakly unilaterally competitive games, weakly competitive games, potential games
JEL Classifications: C72 (Noncooperative games), C73 (Evolutionary games)

1 Introduction

The class of games with weak payoff externalities was introduced by Ania (2008) as a class of symmetric games in which "the effect of any unilateral deviation on the deviator's payoff is always greater than the effect on the opponents' payoffs" (Ania, 2008, p.478). She showed that in a game with weak payoff externalities, a symmetric Nash equilibrium (SNE) is equivalent to a symmetric evolutionary equilibrium (SEE) introduced by Schaffer (1989). This equivalence was generalized by Hehenkamp, Possajennikov, and Guse (2010), who showed that either the weak payoff externalities or their "weak competitiveness" at symmetric profiles is sufficient, and almost necessary, for the equivalence.

An important question arises here: under what conditions does such an equivalent equilibrium *exist*? In this paper, we are concerned with an SNE in pure strategies in finite or

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infinite symmetric games.[1] For infinite symmetric games, an SNE exists if, e.g., the strategy set is a compact and convex subset of a Euclidean space and the payoff function is continuous and quasiconcave (in own strategy) (Moulin, 1986, p.115). Under these standard assumptions, the existence of SNEs, which are equivalent to SEEs, is assured for infinite games with weak payoff externalities or with weak competitiveness.[2] For finite symmetric games, in contrast, the quasiconcavity alone does not guarantee the existence of an SNE in pure strategies.[3] Duersch, Oechssler, and Schipper (2012) showed that there exists an SNE in a two-person symmetric finite zero-sum game if the strategy set is linearly ordered and the payoff function is quasiconcave. Iimura and Watanabe (2016) generalized the result to n-person weakly unilaterally competitive games (Kats and Thisse, 1992) that are finite and symmetric under the same assumptions of linear order and quasiconcavity. It should be noted that a game with weak payoff externalities need not be weakly unilaterally competitive. In particular, there is no two-person zero-sum game that has weak payoff externalities. Hence these results do not apply to games with weak payoff externalities. To our knowledge, however, there is no literature on the existence of Nash equilibria in games with weak payoff externalities. We want to fill this gap.

As a generalization of a game with weak payoff externalities, we introduce the notion of a game with *partial* weak payoff externalities. We say that a game has partial weak payoff externalities if the weak payoff externality condition holds for those opponents whose payoffs increase (resp. decrease) when the deviator's payoff increases (resp. decreases). In fact, the definition requires the condition to be satisfied only for two-strategy subgames of the whole game.[4] Interestingly, the condition reads that "a symmetric profile is an SNE if and only if it is an SEE"; quite an obvious way of securing the equivalence. We show that in a game with partial weak payoff externalities, the set of SNEs coincides with the set of SEEs. Although partial weak payoff externality alone does not guarantee the existence of SNE,[5] we show that if the game is *finite*, the set of SNEs, which are equivalent to SEEs, is nonempty under the assumptions of linear order and quasiconcavity, as in Duersch, Oechssler, and Schipper (2012). The proof invokes a special type of path-acyclicity. Since a weakly unilaterally competitive symmetric game turns out to be a game with partial weak payoff externalities, we now have a unifying class of symmetric games in which SNE and SEE are equivalent and their existence in a finite game is guaranteed by the quasiconcavity on the linearly ordered strategies.

Whereas partial weak payoff externality alone does not guarantee the existence of SNE, it is not clear whether the weak payoff externality condition alone ensures the existence of equilibrium or not. For two-person games, in contrast, we show that the weak payoff externality condition alone is sufficient to guarantee the existence of an SNE. This is shown by a potential function argument.

The rest of the paper is organized as follows. Section 2 introduces the notations and basic definitions. Section 3 provides our main results. Concluding remarks are given in Section 4.

[1] A game is finite if all strategy sets are finite sets; infinite if they are infinite sets.

[2] The same observation applies to a game with partial weak payoff externalities, which we shall introduce shortly. Concerning existence of equilibria, we shall thus mainly focus on finite games.

[3] See, Figure 3 in Section 3.

[4] A restriction embedded in the original definition of weak payoff externalities in Ania (2008) is also removed, i.e., our formulation permits payoff ties.

[5] Consider the Rock-Paper-Scissors game which is not a game with weak payoff externalities. But it *is* a game with partial weak payoff externalities.

2 Preliminaries

We denote by (I_n, S, u) an *n-person symmetric game*, where $I_n = \{1, \dots, n\}$ is a set of players ($n \geq 2$), S is a finite or infinite set ($|S| \geq 2$), and u is a real-valued function on S^n (n-product of S) satisfying

$$u(x, \xi) = u(x, \pi(\xi)) \quad \forall \pi \in \Pi_{n-1},\ \forall (x, \xi) \in S \times S^{n-1},$$

where Π_{n-1} is the set of all permutations of $(n-1)$-tuples. For each player $i \in I_n$, the *strategy set* S_i is given by $S_i = S$ and the *payoff function* u_i is given by

$$u_i(s) = u(\pi_{1i}(s)),$$

where π_{1i} is the identity if $i = 1$, and the transposition of the first and the ith elements if $i \neq 1$. We call the game finite (resp. infinite) if S is finite (resp. infinite).

Let x^k be the k-repetition of x, i.e., $x^k = x, \dots, x$ (k times). A strategy profile $s \in S^n$ is *symmetric* if $s = (x^n)$ for some $x \in S$. If s is symmetric then $u_i(s) = u_j(s)$ for every $i, j \in I_n$. If $s = (x^{i-1}, y, x^{n-i})$, i.e., if player i chooses y and the others x, then $u_i(s) = u(y, x^{n-1})$ and $u_j(s) = u(x^{n-1}, y)$ for every $j \neq i$.

A strategy profile (x^n) is a *symmetric Nash equilibrium* (SNE) if

$$u(x, x^{n-1}) \geq u(y, x^{n-1}) \quad \forall y \in S.$$

An SNE is *strict* if the above inequality is strict. We say that the game is *tie-free* if $u(x, \xi) \neq u(y, \xi)$ for any $x, y \in S$ such that $x \neq y$ and $\xi \in S^{n-1}$, i.e., if there is no payoff tie with respect to the own strategy. If a game is tie-free, then a Nash equilibrium (NE) of a tie-free game is strict. A strategy profile (x^n) is a *weak symmetric evolutionary equilibrium* (Schaffer, 1989) if

$$u(x^{n-1}, y) \geq u(y, x^{n-1}) \quad \forall y \in S.$$

For simplicity, we refer to it as a *symmetric evolutionary equilibrium* (SEE). The definition of an SEE says that if some player changes his strategy from x to y then his payoff never exceeds the payoff of the opponents.

Given a symmetric game $G = (I_n, S, u)$, we call a game $(I_n, \{x, y\}, u)$ a *two-strategy subgame of* G if $x, y \in S$, $x \neq y$, with u restricted to $\{x, y\}^n$.

3 The main results

3.1 The definition and the equivalence of equilibria

A symmetric game (I_n, S, u) is said to have a weak payoff externalities (Ania, 2008) if for any $x, y \in S$ such that $x \neq y$ and $\xi \in S^{n-1}$

$$|u(y, \xi) - u(x, \xi)| > |u(\pi_{1i}(y, \xi)) - u(\pi_{1i}(x, \xi))| \quad \forall i \neq 1. \tag{1}$$

Note that this definition is silent on what should happen if $u(y, \xi) - u(x, \xi) = 0$. A possible interpretation is that the game is being assumed to be tie-free. Let us modify (1) by allowing deviator's payoff ties. Following Marx and Swinkels (1997), we say that a symmetric game (I_n, S, u) satisfies the *transference of decision maker indifference* if

(TDI) for any $x, y \in S$ and $\xi \in S^{n-1}$

$$u(y,\xi) - u(x,\xi) = 0 \implies u(\pi_{1i}(y,\xi)) - u(\pi_{1i}(x,\xi)) = 0 \quad \forall i \neq 1.$$

We say that a symmetric game (I_n, S, u) has a *weak payoff externalities* if

(WPE) for any $x, y \in S$ and $\xi \in S^{n-1}$

$$\left.\begin{array}{l} u(y,\xi) - u(x,\xi) \neq 0 \implies |u(y,\xi) - u(x,\xi)| > |u(\pi_{1i}(y,\xi)) - u(\pi_{1i}(x,\xi))| \\ u(y,\xi) - u(x,\xi) = 0 \implies u(\pi_{1i}(y,\xi)) - u(\pi_{1i}(x,\xi)) = 0 \end{array}\right\} \quad \forall i \neq 1.$$

Note that (TDI) does not place any restriction for tie-free games.

Consider the following condition that generalizes (WPE).

Definition 3.1. A symmetric game (I_n, S, u) has *partial weak payoff externalities* if

(PWPE) for any $x, y \in S$ and $\xi \in \{x, y\}^{n-1}$

$$\left.\begin{array}{l} u(y,\xi) - u(x,\xi) > 0 \implies u(y,\xi) - u(x,\xi) > u(\pi_{1i}(y,\xi)) - u(\pi_{1i}(x,\xi)) \\ u(y,\xi) - u(x,\xi) < 0 \implies u(y,\xi) - u(x,\xi) < u(\pi_{1i}(y,\xi)) - u(\pi_{1i}(x,\xi)) \\ u(y,\xi) - u(x,\xi) = 0 \implies u(\pi_{1i}(y,\xi)) - u(\pi_{1i}(x,\xi)) = 0 \end{array}\right\} \quad \forall i \neq 1.$$

Note that (PWPE) requires (WPE) to hold not for all but for those opponents whose payoffs increase (resp. decrease) when the deviator's payoff increases (resp. decreases), and not in the whole game but in its two-strategy subgames. Note also that the three conditions in (PWPE) are combined to a single bi-conditional:

$$u(y,\xi) - u(x,\xi) > 0 \iff u(y,\xi) - u(x,\xi) > u(\pi_{1i}(y,\xi)) - u(\pi_{1i}(x,\xi)).$$

Therefore, if $\xi = x^{n-1}$, in particular, then $u(y, x^{n-1}) - u(x^n) > 0$ if and only if $u(y, x^{n-1}) > u(x^{n-1}, y)$, i.e.,

$$u(y, x^{n-1}) \leq u(x^n) \iff u(y, x^{n-1}) \leq u(x^{n-1}, y). \tag{2}$$

Varying $y \in S$, this says that (x^n) is an SNE if and only it is an SEE. We thus have:

Proposition 3.1. *For any game satisfying (PWPE), an SNE is equivalent to an SEE.*

3.2 The relationship to competitive games

In what follows, we show how games with partial weak payoff externalities relate to weakly unilaterally competitive games and weak competitive games. A symmetric game (I_n, S, u) is said to be *weakly unilaterally competitive* (Kats and Thisse, 1992)[6] if

(WUC) for any $x, y \in S$ and $\xi \in S^{n-1}$

$$\left.\begin{array}{l} u(y,\xi) > u(x,\xi) \implies u(\pi_{1i}(y,\xi)) \leq u(\pi_{1i}(x,\xi)) \\ u(y,\xi) = u(x,\xi) \implies u(\pi_{1i}(y,\xi)) = u(\pi_{1i}(x,\xi)) \end{array}\right\} \quad \forall i \neq 1.$$

	a	b	c
a	$2, 2$	$0, 3$	$1, -1$
b	$3, 0$	$1, 1$	$1, -1$
c	$-1, 1$	$-1, 1$	$0, 0$

Figure 1: A game satisfying (PWPE) but not (WPE) nor (WUC)

It is straightforward to see that (WUC) implies (PWPE). Hence, the class of games satisfying (PWPE) includes not only games satisfying (WPE), but also games satisfying (WUC). See Figure 1, which shows that these inclusions are indeed strict.

A symmetric game (I_n, S, u) is said to be *weakly competitive* (Hehenkamp, Possajennikov, and Guse, 2010) if

(WC) for any $x, y \in S$ and $\xi \in S^{n-1}$

$$\left.\begin{array}{l} u(y,\xi) \geq u(x,\xi) \implies u(\pi_{1i}(y,\xi)) \leq u(\pi_{1i}(x,\xi)) \\ u(y,\xi) < u(x,\xi) \implies u(\pi_{1i}(y,\xi)) \geq u(\pi_{1i}(x,\xi)) \end{array}\right\} \quad \exists i \neq 1.$$

Letting $\xi = x^{n-1}$, we find that (WC) implies the equivalence of an SNE and an SEE: for any $y \in S$, $u(y, x^{n-1}) \leq u(x^n)$ if and only if $u(y, x^{n-1}) \leq u(x^{n-1}, y)$. Also, it can be shown that (WUC) implies (WC), and (WC) is equivalent to (WUC) in two-person games. However, (WC) does not imply (TDI) if $n > 2$ and $(x, \xi) \neq x^n$. (See Figure 7 of Iimura and Watanabe (2016).) Hence a game satisfying (WC) need not satisfy (PWPE).

See Figure 2 for the relationship among the classes of games.

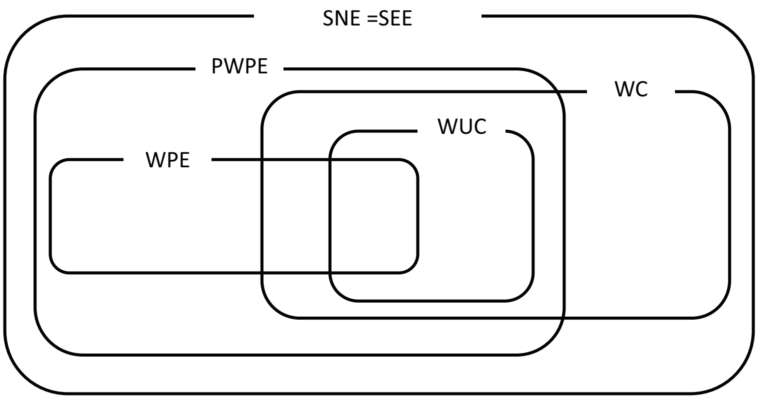

Figure 2: Relationship among the classes of games

3.3 The existence of an SNE in finite games

We first observe that any game satisfying (PWPE) has the following property.

[6]The original definition of a weakly unilaterally competitive game by Kats and Thisse (1992) does not require the game to be symmetric.

Lemma 3.1. *Let $G = (I_n, S, u)$ be a symmetric game that satisfies (PWPE). Then*

$$u(y, x^{n-1}) > u(x^n) \iff u(y^n) > u(x, y^{n-1}) \quad \forall x, y \in S. \tag{$\star$}$$

Proof. Let $x, y \in S$ and $\eta \in \{x, y\}^{n-2}$, where η is void if $n = 2$. We show that

$$u(y, x, \eta) > u(x, x, \eta) \iff u(y, \eta, y) > u(x, \eta, y). \tag{3}$$

Then ($\star$) follows directly if $n = 2$, otherwise by successively substituting η with

$$\eta_0 = (x^{n-2}),\ \eta_1 = (x^{n-3}, y),\ \dots,\ \eta_{n-3} = (x, y^{n-3}),\ \eta_{n-2} = (y^{n-2}),$$

noting for every $k = 0, \dots, n-3$ that $(\eta_k, y) = (x, \eta_{k+1})$ and hence that

$$u(y, \eta_k, y) > u(x, \eta_k, y) \iff u(y, x, \eta_{k+1}) > u(x, x, \eta_{k+1}).$$

Observe that

$$\begin{aligned} u(y, x, \eta) > u(x, x, \eta) &\iff u(y, x, \eta) - u(x, x, \eta) > u(x, y, \eta) - u(x, x, \eta) \quad \text{by (PWPE)} \\ &\iff u(x, y, \eta) - u(y, y, \eta) < u(y, x, \eta) - u(y, y, \eta) \\ &\iff u(x, y, \eta) < u(y, y, \eta) \quad \text{by (PWPE)}. \end{aligned} \tag{4}$$

Hence $u(y, x, \eta) > u(x, x, \eta) \iff u(y, y, \eta) > u(x, y, \eta)$, which is equivalent to (3) by symmetry. □

In the sequel, we assume that S is a finite and *linearly ordered* set. For $a, b, c \in S$, we say that b is between a and c if $\min\{a, c\} < b < \max\{a, c\}$.

Definition 3.2 (Duersch, Oechssler, and Schipper (2012))**.** The payoff function is *quasiconcave* (in own strategy) if

(QC) for any $x, x', x'', y \in S$ such that x is between x' and x'',

$$u(x, y^{n-1}) \geq \min\{u(x', y^{n-1}), u(x'', y^{n-1})\}.$$

G is *quasiconcave* if the payoff function is quasiconcave.

Remark 3.1. For infinite symmetric games, an SNE exists if, e.g., the strategy set is a compact and convex subset of a Euclidean space and the payoff function is continuous and quasiconcave (in own strategy) (Moulin, 1986, p.115). For finite symmetric games, in contrast, the quasiconcavity alone does not guarantee the existence of an SNE in pure strategies. A game in Figure 3 is an example.

Remark 3.2. In Duersch, Oechssler, and Schipper (2012), (QC) is defined for two-person games. In Iimura and Watanabe (2016), it is defined for n-person games, given any profile of the opponents' strategies in S^{n-1}. The current definition, which assumes $y^{n-1} \in S^{n-1}$, is slightly weaker.

We now show:

	1	2	3
1	$-1, -1$	$1, 0$	$0, 1$
2	$0, 1$	$0, 0$	$1, -1$
3	$1, 0$	$-1, 1$	$-1, -1$

Figure 3: A game satisfying (QC) does not have an equilibrium

Proposition 3.2. *Let $G = (I_n, S, u)$ be a finite symmetric game that satisfies (PWPE). If S is linearly ordered and u satisfies (QC), then G has an SNE. If in addition G is tie-free then the SNE is unique.*

Proof. Let $\beta(x) = \{y \in S \mid u(y, x^{n-1}) \geq u(z, x^{n-1})\ \forall z \in S\}$ for each $x \in S$, and consider the sequence $(x_k)_{k\geq 0}$ of S generated by a rule:

> (BR) For $k = 0$, pick an arbitrary $x_0 \in S$. For $k \geq 1$, pick $x_{k+1} \in \beta(x_k)$ if there exists some $y \in \beta(x_k)$ such that $u(y, x_k^{n-1}) > u(x_k^n)$; otherwise terminate at x_k.

We show that for any such sequence there exists a natural number $T \geq 0$ such that the sequence terminates at $x_T \in S$. That $(x_T^n) \in S^n$ is an SNE is obvious by the construction. Since S is finite, it suffices to show that the sequence never includes a cycle. Now, suppose by way of contradiction that the sequence includes a cycle $x_0, x_1, \ldots, x_p = x_0$ $(p \geq 2)$, whose underlying set is denoted by C. Let $a = \min C$ and $b = \max C$. Then there exists $x \in C$ such that $a < x \leq b$ and $u(a, x^{n-1}) > u(x^n)$, and $y \in C$ such that $a \leq y < b$ and $u(b, y^{n-1}) > u(y^n)$. By $(\star)$ of Lemma 3.1, which is an implication of (PWPE), $u(a, x^{n-1}) > u(x^n)$ implies $u(a^n) > u(x, a^{n-1})$, and $u(b, y^{n-1}) > u(y^n)$ implies $u(b^n) > u(y, b^{n-1})$. By (QC), $u(a^n) > u(x, a^{n-1})$ implies $u(a^n) > u(b, a^{n-1})$, and $u(b^n) > u(y, b^{n-1})$ implies $u(b^n) > u(a, b^{n-1})$. However, that $u(a^n) > u(b, a^{n-1})$ and $u(b^n) > u(a, b^{n-1})$ contradicts $(\star)$. Hence there exists some $T \geq 0$ such that the sequence terminates at x_T.

To see the last claim, suppose that G is tie-free. Then $(\star)$ immediately implies the uniqueness of the SNE. □

3.4 Two-person games

In this subsection, we consider a *two-person* symmetric game satisfying (WPE), as opposed to (PWPE). The set S of strategies need not be finite nor ordered. Hence the game may not be quasiconcave.

Let P be a function defined on S^2 by $P(x, y) = u(x, y) + u(y, x)$, the payoff sum of the players. If $u(y, z) \neq u(x, z)$ then (WPE) says that $|u(y, z) - u(x, z)| > |u(z, y) - u(z, x)|$, so $u(y, z) > u(x, z)$ implies $u(y, z) - u(x, z) > u(z, x) - u(z, y)$, i.e., $u(y, z) + u(z, y) > u(x, z) + u(z, x)$. Thus $u(y, z) > u(x, z)$ implies $P(y, z) > P(x, z)$. Changing the role of y and x, $u(y, z) < u(x, z)$ implies $P(y, z) < P(x, z)$. In addition, the (TDI) condition stipulates that $u(y, z) = u(x, z)$ implies $P(y, z) = P(x, z)$. Hence a two-person symmetric game satisfying (WPE) is an ordinal potential game (Monderer and Shapley, 1996) in which the payoff sum P works as its ordinal potential function:[7]

$$u(y,z) > u(x,z) \iff P(y,z) > P(x,z) \quad \forall x,y,z \in S.$$

Hence, with appropriate topological condition, there exists an NE. Actually, an *SNE* exists.

Proposition 3.3. *Let $G = (I_2, S, u)$ be a two-person symmetric game satisfying (WPE). Assume also that S is a compact topological space and u is continuous on S^2 endowed with product topology if S is infinite. Then G has an SNE. If in addition G is tie-free then it is a unique NE of G.*

Proof. As we have observed, G has an ordinal potential function $P\colon S^2 \to \mathbb{R}$ defined by $P(x,y) = u(x,y) + u(y,x)$. If u is continuous and S is compact, then P is continuous and has a maximum on the compact set S^2. Any maximizer of P is a NE. Let $(y,x) \in S^2$ be a NE. Since (WPE) implies (PWPE), Lemma 3.1 applies, and the condition ($\star$) reads as follows:

$$u(y,x) > u(x,x) \iff u(y,y) > u(x,y). \tag{5}$$

Now, $u(y,x) \geq u(x,x)$ by equilibrium condition. By (5),

$$u(y,y) \geq u(x,y).$$

By symmetry, $(x,y) \in S^2$ is another NE. By equilibrium condition,

$$u(y,y) \leq u(x,y).$$

Hence $u(y,y) = u(x,y)$, so $u(y,y) = u(x,y) \geq u(z,y)$ for all $z \in S$, i.e., $u_1(y,y) \geq u_1(z,y)$ for all $z \in S$ and $u_2(y,y) \geq u_2(y,z)$ for all $z \in S$ by symmetry. Hence (y,y) is an SNE.

To see the last claim, suppose that G is tie-free and (x,y) is a NE for some $x \neq y$. Then (y,x) is also a NE by symmetry. Since G is tie-free, we have $u(x,y) > u(y,y)$ and $u(y,x) > u(x,x)$. It contradicts (5). Hence every NE must be symmetric if G is tie-free. Also, it is unique by (5). □

4 Concluding remarks

Let $x,y \in S$ and $\eta \in \{x,y\}^{n-2}$. In a game satisfying (PWPE), $u(y,x,\eta) > u(x,y,\eta)$ implies $u(y,y,\eta) > u(x,y,\eta)$ by (4). The first inequality says that the x-player may have a incentive to imitate the y-player. The second inequality says that the imitation results in a payoff improvement. It follows that "imitating the better" is individually improving under (PWPE). For a finite game, we can conceive of a sequence of symmetric profiles such that for each step a unilateral deviation governed by (BR) is followed by $n-1$ imitations, leading to another symmetric profile. This process need not be Pareto improving. However, if the game is a two-person tie-free game with (WPE), then the resulting unique SNE, which is also an SEE, is Pareto efficient.

[7]Recall that (WPE) is a conjunction of (1) and (TDI). One can verify that a two-person game satisfying (1) is a *generalized ordinal potential game* (Monderer and Shapley, 1996) in which the payoff sum P works as its generalized ordinal potential function.

Let us recapitulate some crucial conditions and their roles. For n-person games, (PWPE) and (QC) are the key conditions. (PWPE) is implied by (WPE) or (WUC), and implies ($\star$) and (2). Concerning the existence of equilibrium, we make use of ($\star$) and (QC). Concerning the equivalence between SNE and SEE, all we need is (2) at symmetric strategy profiles. Although both conditions are implied by (PWPE), ($\star$) and (2) are independent: Neither implies the other.

For two-person games, there is a single key condition, (WPE). (PWPE) is too weak to let the potential argument work. Needless to say, the argument based on (PWPE) and (QC) applies in a two-person game as well.

Finally, let us go back to condition ($\star$). In a two-person symmetric game, it reduces to the condition that $u(y, x) > u(x, x)$ if and only if $u(y, y) > u(x, y)$ for any $x, y \in S$. In Iimura, Maruta, and Watanabe (2016), we call a two-person symmetric game *pairwise solvable* if it satisfies this condition. Investigating two-person pairwise solvable games, results in Iimura, Maruta, and Watanabe (2016) include sets of conditions for the existence of equilibrium. It turns out that these existence results may be applied to prove the existence of SNE in n-person symmetric games. Given an n-person symmetric game $G = (I_n, S, u)$, let $\tau(G) = (I_2, S, v)$ be a two-person symmetric game such that

$$v(x, y) = u(x, y^{n-1}).$$

It can be verified that for any $x \in S$, $x^2 \in S^2$ is an SNE in $\tau(G)$ if and only if $x^n \in S^n$ is an SNE in G. Moreover, if G satisfies ($\star$) then $\tau(G)$ is pairwise solvable. Consequently, the existence problem in an n-person symmetric game with partial weak payoff externalities comes down to that of the associated two-person pairwise solvable game.

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